NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2522

A GRAPHICAL METHOD FOR PLOTTING AMPLITUDE AND PHASE ANGLE OF TRANSFER FUNCTIONS OF DYNAMIC SYSTEMS WITHOUT FACTORING POLYNOMIALS

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A GRAPHICAL METHOD FOR PLOTTING AMPLITUDE AND PHASE ANGLE OF TRANSFER FUNCTIONS OF DYNAMIC SYSTEMS WITHOUT FACTORING POLYNOMIALS

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SUMMARY

A method is presented for obtaining amplitude and angle plots for rational algebraic functions of an imaginary variable. Application of the method is illustrated by an example in which the frequency response of an automatically controlled aircraft is plotted. The method involves the use of templets but does not require the factoring of polynomials. Amplitude and angle plots of high-degree rational functions of an imaginary variable can be obtained more rapidly with this method than by analytical calculation or by methods involving the factoring of polynomials.

INTRODUCTION

If a dynamic system can be represented by a set of linear differential equations with constant coefficients and with time as the independent variable, the relationship between any output variable of the system θ_o and any input of a system θ_i can be obtained in the operational form

$$\frac{\theta_o}{\theta_i}(s) = K \frac{s^p + a_{p-1}s^{p-1} + \ . \ . \ . + a_0}{s^q + b_{q-1}s^{q-1} + \ . \ . \ . + b_0}$$

where s denotes the operation $\frac{d}{dt}$ and $K, a_0, a_1 \ . \ . \ ., b_0, b_1 \ . \ . \ .,$ p, and q are constants. This function is usually referred to as the transfer function of the system. It is shown in chapter 4 of reference 1 that when s is replaced by $j\omega$, where $j = \sqrt{-1}$, the resulting complex function of ω represents the frequency response of the system.

That is, when any value of forcing frequency ω, in radians per second, is substituted into the function, the amplitude and angle of the resulting complex value of the function represent the amplification and phase shift, respectively, through the system when excited at that frequency.

In the analysis and synthesis of such systems, it is often desirable to obtain plots of the amplification or amplitude A and the phase shift or phase angle φ for the system over a portion of the frequency spectrum. These plots can be obtained by evaluating the transfer function of $j\omega$ at discrete values of ω and plotting amplitude and angle as functions of ω, but this process is tedious and time consuming. Another method of obtaining the plots is to factor the numerator and denominator of the transfer function into first-degree and second-degree factors and combine them graphically by using templets to represent each factor on logarithmic coordinates as described in reference 1. But, when high-degree polynomials are involved, the process of factoring becomes very tedious.

The present method allows the use of templets for plotting rational functions of $j\omega$ but does not require the factoring of polynomials. The basic principle involved is the use of logarithms (or logarithmic graph paper) to simplify the plotting of the individual terms of the real and imaginary parts of the numerator and denominator polynomials. This principle was initially conceived and applied to rational functions by Mr. William M. Kauffman of NACA Headquarters. Most of the templets used, in the present method, to perform the graphical operations required in obtaining the frequency plots were first developed and used by Mr. Kauffman.

SYMBOLS

θ_o	output variable of a system
θ_i	input (excitation) of a system
s	transform variable corresponding to differential operator $\frac{d}{dt}$
$j = \sqrt{-1}$	
ω	excitation frequency, radians per second

u	excitation frequency, k-radians per second (ω/k)
$K,\ a_0, a_1 \ldots,\ b_0, b_1 \ldots,\ p,\ q$	constants

$F(j\omega) = A(\omega)e^{j\varphi(\omega)} = R(\omega) + jI(\omega)$

$A(\omega)$	amplitude of $F(j\omega)$
$\varphi(\omega)$	angle of $F(j\omega)$
$I(\omega)$	imaginary part of $F(j\omega)$
$R(\omega)$	real part of $F(j\omega)$

THEORETICAL ANALYSIS

A general rational algebraic function of $j\omega$ can be written in the form

$$F(j\omega) = K\frac{(j\omega)^p + \ldots + a_5(j\omega)^5 + a_4(j\omega)^4 + a_3(j\omega)^3 + a_2(j\omega)^2 + a_1(j\omega) + a_0}{(j\omega)^q + \ldots + b_5(j\omega)^5 + b_4(j\omega)^4 + b_3(j\omega)^3 + b_2(j\omega)^2 + b_1(j\omega) + b_0}$$

$$= K\frac{(j\omega)^p + \ldots + a_5\omega^5 j + a_4\omega^4 - a_3\omega^3 j - a_2\omega^2 + a_1\omega j + a_0}{(j\omega)^q + \ldots + b_5\omega^5 j + b_4\omega^4 - b_3\omega^3 j - b_2\omega^2 + b_1\omega j + b_0}$$

where K, $a_0, a_1 \ldots$, $b_0, b_1 \ldots$, p, and q are constants.

The function could also be written in the following forms:

$$F(j\omega) = K \frac{\left[R(\omega)\right]_N + j\left[I(\omega)\right]_N}{\left[R(\omega)\right]_D + j\left[I(\omega)\right]_D}$$

$$= K \frac{\left[A(\omega)\right]_N e^{j\left[\varphi(\omega)\right]_N}}{\left[A(\omega)\right]_D e^{j\left[\varphi(\omega)\right]_D}}$$

$$= K\,A(\omega)e^{j\varphi(\omega)}$$

where

$$\left[R(\omega)\right]_N = a_0 - a_2\omega^2 + a_4\omega^4 + \ldots = \text{Real part of numerator}$$

$$\left[I(\omega)\right]_N = a_1\omega - a_3\omega^3 + a_5\omega^5 + \ldots = \text{Imaginary part of numerator}$$

$$\left[R(\omega)\right]_D = b_0 - b_2\omega^2 + b_4\omega^4 + \ldots = \text{Real part of denominator}$$

$$\left[I(\omega)\right]_D = b_1\omega - b_3\omega^3 + b_5\omega^5 + \ldots = \text{Imaginary part of denominator}$$

$$\left[A(\omega)\right]_N = \sqrt{\left[R(\omega)\right]_N^2 + \left[I(\omega)\right]_N^2} = \text{Amplitude of numerator}$$

$$\left[\varphi(\omega)\right]_N = \tan^{-1}\frac{\left[I(\omega)\right]_N}{\left[R(\omega)\right]_N} = \text{Angle of numerator}$$

$$\left[A(\omega)\right]_D = \sqrt{\left[R(\omega)\right]_D^2 + \left[I(\omega)\right]_D^2} = \text{Amplitude of denominator}$$

$$\left[\varphi(\omega)\right]_D = \tan^{-1} \frac{\left[I(\omega)\right]_D}{\left[R(\omega)\right]_D} = \text{Angle of denominator}$$

$$A(\omega) = \frac{\left[A(\omega)\right]_N}{\left[A(\omega)\right]_D} = \text{Amplitude of the rational function}$$

$$\varphi(\omega) = \left[\varphi(\omega)\right]_N - \left[\varphi(\omega)\right]_D = \text{Angle of the rational function}$$

The real and imaginary parts of either the numerator or the denominator can be obtained by algebraic addition of the individual terms each of which is a function of ω. The logarithm of a typical term $c\omega^n$ is

$$\log c\omega^n = \log c + n(\log \omega)$$

but $\log c$ is a constant and $n(\log \omega)$ is a linear function of $\log \omega$. Therefore, each term of the real or imaginary part of numerator or denominator is a straight line of slope n when plotted logarithmically as a function of ω, that is, when the logarithm of the function is plotted against the logarithm of ω on a linear scale or, equivalently, when the function is plotted against ω on a logarithmic scale. (This type of plot is hereinafter referred to as a logarithmic plot.) Although a logarithmic plot of each of the separate terms is very easy to make, these terms must be added (or subtracted), and graphical addition is not a simple operation when the curves to be added are plotted logarithmically. But it is possible to make a templet which can be used to add (or subtract) graphically any two values plotted logarithmically. Such a templet is shown in figure 1. Its construction is described in the appendix and its practical use is described in a subsequent section entitled "Practical Application of Methods and Description of Templets." By use of such a templet, any two logarithmic curves can be added or subtracted point by point.

Consider any two functions of ω, $F_1(\omega)$ and $F_2(\omega)$, having values P and Q, respectively, at $\omega = \omega_1$ and plotted logarithmically as shown in figure 1. Then

$$\begin{aligned}\log\left[F_1(\omega_1) + F_2(\omega_1)\right] &= \log(P + Q) \\ &= \log Q\left(1 + \frac{P}{Q}\right) \\ &= \log Q + \log\left(1 + \frac{P}{Q}\right)\end{aligned}$$

Since the distance between P and Q on a logarithmic plot depends only on the ratio P/Q and not on the individual values of P and Q, a templet can be made which converts this distance corresponding to $\log \frac{P}{Q}$ into a distance corresponding to $\log\left(1 + \frac{P}{Q}\right)$, which is also dependent only on the ratio P/Q, and adds this latter distance to the point representing log Q to locate the point representing log(P + Q). Similarly,

$$\log(P - Q) = \log P + \log\left(1 - \frac{Q}{P}\right) \qquad (P > Q)$$

and a templet can be made which converts a distance corresponding to $\log \frac{Q}{P}$ to a distance corresponding to $\log\left(1 - \frac{Q}{P}\right)$ and adds the latter distance to the point representing log P to locate the point representing log(P - Q). Such a templet is shown in figure 1 and is hereinafter referred to as the summing templet.

If the logarithmic plots of the functions $F_1(\omega)$ and $F_2(\omega)$ are straight lines, as in the case of the individual terms of the real and imaginary parts of numerator or denominator of $F(j\omega)$, the shape of the logarithmic plot of their sum or difference can be shown to depend only on the slopes of the two straight lines. That is, shifting either or both of the straight lines in any direction merely shifts their point of intersection and hence shifts the curve representing their sum or difference but does not alter the shape of the curve. Therefore, templets can be made representing the sum and difference of pairs of straight lines of various slopes. Each of these templets when properly located can give directly a logarithmic plot of the sum or difference of a pair of terms of the real or imaginary part of the numerator or denominator of $F(j\omega)$. The contours for four such templets, which are sufficient for plotting polynomials of ninth degree or less, are shown in figure 2 and these templets are hereinafter referred to as term-pair templets or term-pair contours.

After logarithmic plots of the real and imaginary parts of numerator or denominator have been obtained, it is necessary to obtain the corresponding amplitude and angle. The amplitude can be obtained graphically by use of a templet similar to the summing templet since

$$\log A(\omega) = \log \sqrt{\left[R(\omega)\right]^2 + \left[I(\omega)\right]^2}$$

$$= \log R(\omega) \sqrt{1 + \left[\frac{I(\omega)}{R(\omega)}\right]^2}$$

$$= \log R(\omega) + \log \sqrt{1 + \left[\frac{I(\omega)}{R(\omega)}\right]^2}$$

or

$$\log A(\omega) = \log \sqrt{\left[R(\omega)\right]^2 + \left[I(\omega)\right]^2}$$

$$= \log I(\omega) \sqrt{1 + \left[\frac{R(\omega)}{I(\omega)}\right]^2}$$

$$= \log I(\omega) + \log \sqrt{1 + \left[\frac{R(\omega)}{I(\omega)}\right]^2}$$

A templet can be made which at any value of frequency ω_1 converts the distance corresponding to $\log \frac{I(\omega_1)}{R(\omega_1)}$ $\left(\text{or} \quad \log \frac{R(\omega_1)}{I(\omega_1)}\right)$ into a distance corresponding to $\log\sqrt{1 + \left[\frac{I(\omega_1)}{R(\omega_1)}\right]^2}$ $\left(\text{or} \quad \log\sqrt{1 + \left[\frac{R(\omega_1)}{I(\omega_1)}\right]^2}\right)$ and adds this distance to the point representing $\log R(\omega_1)$ $\left(\text{or} \quad \log I(\omega_1)\right)$ to locate the point representing $\log\sqrt{\left[R(\omega_1)\right]^2 + \left[I(\omega_1)\right]^2}$. Such a templet is shown in figure 3(a) and is hereinafter referred to as the amplitude templet.

Since the angle $\tan^{-1}\frac{I(\omega)}{R(\omega)}$ is dependent only upon the ratio $I(\omega)/R(\omega)$, it can be obtained graphically at any value of frequency ω_1 from the logarithmic plots of $I(\omega)$ and $R(\omega)$ by use of a scale which converts the distance corresponding to $I(\omega_1)/R(\omega_1)$ (i.e., the distance between $I(\omega_1)$ and $R(\omega_1)$ on a logarithmic plot) into angle directly. Such a scale is shown in figure 3(b) and is hereinafter referred to as the angle scale

After obtaining the amplitudes $\left[A(\omega)\right]_N$ and $\left[A(\omega)\right]_D$ and the angles $\left[\varphi(\omega)\right]_N$ and $\left[\varphi(\omega)\right]_D$, it is then necessary to combine them to obtain the amplitude $A(\omega)$ and the angle $\varphi(\omega)$ of the original function

$$F(j\omega) = KA(\omega)e^{j\varphi(\omega)} = K\frac{\left[A(\omega)\right]_N}{\left[A(\omega)\right]_D}e^{j\left\{\left[\varphi(\omega)\right]_N - \left[\varphi(\omega)\right]_D\right\}}$$

The angle is obtained by merely subtracting $\left[\varphi(\omega)\right]_D$ from $\left[\varphi(\omega)\right]_N$ for any value of frequency ω_1 after $\left[\varphi(\omega)\right]_D$ and $\left[\varphi(\omega)\right]_N$ have been obtained by use of the angle scale.

By taking logarithms of both sides of the relation $A(\omega) = \frac{\left[A(\omega)\right]_N}{\left[A(\omega)\right]_D}$ the following relation is determined:

$$\log A(\omega) = \log\left[A(\omega)\right]_N - \log\left[A(\omega)\right]_D = \log\left[A(\omega)\right]_N + \log\frac{1}{\left[A(\omega)\right]_D}$$

Therefore, a logarithmic plot of $A(\omega)$ can be obtained by graphically subtracting, point-by-point, the logarithmic plots of $\left[A(\omega)\right]_N$ and $\left[A(\omega)\right]_D$ or by adding the logarithmic plots of $\left[A(\omega)\right]_N$ and $\frac{1}{\left[A(\omega)\right]_D}$.

Since $\log 1 = 0$, the axis representing unity on the logarithmic plots must be used as the reference or zero axis when adding or subtracting distances on a logarithmic plot. The logarithmic plot of $1/f(\omega)$ is merely the logarithmic plot of $f(\omega)$ reflected in the axis representing unity.

PRACTICAL APPLICATION OF METHOD AND DESCRIPTION OF TEMPLETS

The general form of either the numerator or denominator polynomial of a rational algebraic function of $(j\omega)$, $F(j\omega) = \frac{P_1(j\omega)}{P_2(j\omega)}$, is

$$P(j\omega) = a_0 + a_1(j\omega) + a_2(j\omega)^2 + a_3(j\omega)^3 + a_4(j\omega)^4 + a_5(j\omega)^5 + a_6(j\omega)^6 + a_7(j\omega)^7 + \dots$$

$$= a_0 + ja_1\omega - a_2\omega^2 - ja_3\omega^3 + a_4\omega^4 + ja_5\omega^5 - a_6\omega^6 - ja_7\omega^7 + \dots$$

where $a_0, a_1, a_2, \dots$, are constants. In the practical case, the variable ω is usually excitation frequency. In certain cases, it may be necessary to factor $(j\omega)^n$ from the polynomial to obtain a constant term in the remaining factor as in the above form and to correct later for the factor $(j\omega)^n$. This point is mentioned again subsequently.

The real and imaginary parts of $P(j\omega)$ are

$$\text{Real part} = R(\omega) = \left(a_0 - a_2\omega^2\right) + \left(a_4\omega^4 - a_6\omega^6\right) + \dots$$

$$\text{Imaginary part} = I(\omega) = \left(a_1\omega - a_3\omega^3\right) + \left(a_5\omega^5 - a_7\omega^7\right) + \dots$$

and term-pair templets can be made to represent each of the term-pairs enclosed by parentheses. The contours of such templets are shown in figure 2.

In order to get either $R(\omega)$ or $I(\omega)$, the proper term-pairs are plotted by use of the term-pair templets and are summed by use of the summing templet. In order to use the term-pair templets, the intersection of the straight lines representing the logarithmic plots of the separate terms is first located. This intersection fixes the position of the term-pair templet so that its contours represent the logarithmic plot of the sum and difference of the pair of terms. If $a_0, a_1, a_2, \ldots$ are all of the same algebraic sign, the terms of $R(\omega)$ and $I(\omega)$ will have alternating algebraic signs so that term-pairs composed of consecutive terms will always be the difference of two terms rather than the sum.

It may be noted here that logarithmic plots can represent only absolute (or positive) values. It is apparent that for large values of ω, the expression $a_0 - a_2\omega^2$, for example, will be negative (assuming a_0 and a_2 to be positive). For these values of ω, a logarithmic plot must represent $a_2\omega^2 - a_0$ or $-\left(a_0 - a_2\omega^2\right)$ instead of $a_0 - a_2\omega^2$. When graphically obtaining the sum or difference of two functions $F_1(\omega)$ and $F_2(\omega)$, it is necessary to know whether the plot represents $F(\omega)$ or $-F(\omega)$. If then, for example, a logarithmic plot of $F_1(\omega) + F_2(\omega)$ is to be obtained and logarithmic plots of $F_1(\omega)$ and $-F_2(\omega)$ have been obtained, $F_1(\omega) + F_2(\omega)$ may be found by graphically subtracting $-F_2(\omega)$ from $F_1(\omega)$ since $F_1(\omega) + F_2(\omega) = F_1(\omega) - \left[-F_2(\omega)\right]$.

After two logarithmic curves (such as term-pair contours) are drawn, their sum or difference can be obtained graphically by use of the summing templet of figure 1. This templet is used to add two logarithmic curves $F_1(\omega)$ and $F_2(\omega)$ graphically as follows:

The reference point X (a small hole in the templet) is set on the lower curve at some value of frequency ω_1, and contour a is set on the upper curve at ω_1 as illustrated in figure 1. The point representing the sum of the two curves at ω_1 is found on contour b at ω_1. This process is repeated at as many values of ω as are needed to define the resulting logarithmic plot.

In order to subtract two logarithmic curves, the reference point X is set on the upper curve at some value of frequency ω_1 and contour c (or contour d) is set on the lower curve at ω_1. The point representing the difference of the two curves at ω_1 is found on contour d (or contour c) at ω_1. This point represents the upper curve minus the lower curve or the negative of the lower curve minus the upper curve at ω_1. Repeat at as many values of ω as are needed to define the resulting logarithmic plot.

This process of addition and subtraction of curves is continued until all terms of $R(\omega)$ or $I(\omega)$ have been included and logarithmic plots of $R(\omega)$ and $I(\omega)$ have thus been obtained. Then $\sqrt{\left[I(\omega)\right]^2 + \left[R(\omega)\right]^2}$ can be obtained graphically by applying the amplitude templet of figure 3(a) to the logarithmic plots of $R(\omega)$ and $I(\omega)$ as follows:

The reference point Y of the amplitude templet (represented by a small hole in the templet) is set on the lower curve at some frequency ω_1, and contour e is set on the upper curve at ω_1. The point representing $\sqrt{\left[I(\omega)\right]^2 + \left[R(\omega)\right]^2}$ at ω_1 is on contour f at ω_1. This process is repeated at as many values of ω as are needed to define the resulting logarithmic plot of $\sqrt{\left[I(\omega)\right]^2 + \left[R(\omega)\right]^2}$.

The angle scale of figure 3(b) is used in conjunction with logarithmic plots of $R(\omega)$ and $I(\omega)$ to obtain values of $\tan^{-1} \frac{I(\omega)}{R(\omega)}$. The line r near the middle of the scale is placed on the logarithmic plot of $R(\omega)$ at any frequency value ω_1. The angle $\tan^{-1} \frac{I(\omega_1)}{R(\omega_1)}$ is read directly in degrees from the scale at the point representing $I(\omega_1)$. The quadrant of the angle is determined from the algebraic signs of I and R as indicated on the angle scale. This process is repeated for as many values of ω as are needed. The angle values thus obtained may be either tabulated or plotted as a function of ω.

The remaining step is the combination of numerator and denominator to obtain the function

$$F(j\omega) = KA(\omega)e^{j\varphi(\omega)}$$

$$= K\frac{\left[A(\omega)\right]_N}{\left[A(\omega)\right]_D}e^{j\left\{\left[\varphi(\omega)\right]_N - \left[\varphi(\omega)\right]_D\right\}}$$

The angle $\varphi(\omega)$ may be obtained by point-by-point subtraction of $\left[\varphi(\omega)\right]_D$ from $\left[\varphi(\omega)\right]_N$ either graphically or analytically. The logarithmic plot of $A(\omega)$ may be obtained by point-by-point graphical addition of $\log\left[A(\omega)\right]_N$ and $\log\frac{1}{\left[A(\omega)\right]_D}$ $\left(\text{or } \log\left[A(\omega)\right]_N - \log\left[A(\omega)\right]_D\right)$.

If $(j\omega)^n$ was factored from either the numerator or denominator polynomial, it must be included at this point. That is, the amplitude ω^n and the angle, $n \times 90^\circ$ must be added to or subtracted from the amplitude and angle curves already obtained.

For a polynomial $P(j\omega) = a_0 + a_1(j\omega) + a_2(j\omega)^2 + \ldots,$ the values of the coefficients $a_0, a_1, a_2 \ldots$ may be of such widely differing magnitudes as to make the plotting of the various terms on the same graph impractical. This situation may be remedied by a change in frequency scale $\omega = ku$. This and the other foregoing procedures are illustrated in the section entitled "Illustrative Example."

Illustrative Example

A transfer function representative of those encountered in physical systems is

$$F(s) = \frac{731\left(s^3 + 91.5s^2 + 8060s + 26850\right)}{s^5 + 93.5s^4 + 9110s^3 + 273000s^2 + 7550000s + 19630000}$$

This transfer function was encountered in analyzing an idealized system made up of an airframe and automatic control system which controlled the pitch attitude of the airframe.

If s is replaced by $j\omega$ where ω is excitation frequency and j is $\sqrt{-1}$, the resulting function of $j\omega$ represents the steady-state frequency response of the system (reference 1, p. 96)

$$F(j\omega) = \frac{731\left[(j\omega)^3 + 91.5(j\omega)^2 + 8060(j\omega) + 26850\right]}{(j\omega)^5 + 93.5(j\omega)^4 + 9110(j\omega)^3 + 273000(j\omega)^2 + 7550000(j\omega) + 19630000}$$

The magnitudes of the coefficients of the separate terms differ greatly, the ratio of the largest and smallest being 19,630,000. This makes the plotting of the various terms on the same graph inconvenient. However, if a change in frequency scale $\omega = 40u$ is made, the resulting expression is

$$F(ju) = \frac{0.457\left[(ju)^3 + 2.29(ju)^2 + 5.04(ju) + 0.42\right]}{(ju)^5 + 2.34(ju)^4 + 5.69(ju)^3 + 4.27(ju)^2 + 2.95(ju) + 0.192}$$

The ratio of the largest and smallest coefficients in this expression is $\frac{5.69}{0.192} = 29.6$. The factor 0.457 being ignored for the present, the real and imaginary parts of numerator and denominator are

$$\left[R(u)\right]_N = \left(0.42 - 2.29u^2\right)$$

$$\left[I(u)\right]_N = \left(5.04u - u^3\right)$$

$$\left[R(u)\right]_D = \left(0.192 - 4.27u^2\right) + 2.34u^4$$

$$\left[I(u)\right]_D = \left(2.95u - 5.69u^3\right) + u^5$$

The contours of the two templets of figures 2(a) and 2(b) can be used to represent the term-pairs in parentheses. Since the numerator is only of third degree, its real and imaginary parts can each be represented directly by one of the term-pair templet contours. These contours are shown properly orientated on the graph in figure 4, and the real and imaginary parts are combined by use of the amplitude templet to obtain

$\left[A(u)\right]_N = \sqrt{\left[I(u)\right]_N^{\,2} + \left[R(u)\right]_N^{\,2}}$ at any value of u. Figure 4 illustrates

the use of the amplitude templet to obtain $\sqrt{(0.365)^2 + (0.755)^2} = 0.84$ at $u = 0.15$. The angle $\left[\varphi(u)\right]_N = \tan^{-1} \dfrac{\left[I(u)\right]_N}{\left[R(u)\right]_N}$ is obtained from the logarithmic plots of the real and imaginary parts by use of the angle scale. Figure 4 illustrates the use of the angle scale to obtain $\tan^{-1} \dfrac{4.03}{-1.84} = 114.5^{\circ}$ at $u = 1$. The angle values are tabulated for discrete values of u in table I.

Since the denominator of $F(ju)$ is of fifth degree, an extra term must be added to a term-pair to obtain either the real or imaginary part of the denominator. This addition is performed in figure 5 for $\left[R(u)\right]_D$ and in figure 6 for $\left[I(u)\right]_D$. Figure 5 illustrates the use of the summing templet for graphically subtracting $2.5 - 0.9 = 1.6$. Figure 6 illustrates the use of the summing templet for graphically adding $0.120 + 0.335 = 0.455$. The amplitude templet and the angle scale are then applied to the logarithmic plots of $\left[R(u)\right]_D$ and $\left[I(u)\right]_D$ to obtain $\left[A(u)\right]_D$ and $\left[\varphi(u)\right]_D$ shown in figure 7 and table I, respectively. The operations of figures 4, 5, 6, and 7 could be carried out on a single graph, but they have been separated here for clarity. Values of $\left[\varphi(u)\right]_D$ are tabulated in table I for discrete values of u.

The subtraction of $\left[\varphi(u)\right]_D$ from $\left[\varphi(u)\right]_N$ yields total angle $\varphi(u)$ which is also tabulated in table I. If $\left[A(u)\right]_N$ and $\dfrac{1}{\left[A(u)\right]_D}$ are plotted on the same graph, they can be combined to obtain $A(u)$ as in figure 8. This operation consists of the addition of the graphical distances measured vertically from the horizontal axis representing unity (since $\log 1 = 0$) to the logarithmic plots of $\boxed{A(u)}_N$ and $\frac{1}{\boxed{A(u)}_D}$. Distances measured upward from the unity axis are considered positive while those measured downward from the unity axis are considered negative. The factor 0.457 of $F(ju)$, which has been ignored until now, has been taken into account at this point by shifting the logarithmic plot of $\boxed{A(u)}_N$ vertically an amount corresponding to $\log 0.457$. Plots of $A(u)$ and $\varphi(u)$ obtained by the present method are compared with the analytically calculated plots in figures 8 and 9.

General Comments

Because of the steep slopes of the curves being added and subtracted and because this method frequently involves the graphical subtraction of two large, nearly equal quantities, care in performing the graphical operations is necessary in order to maintain a high degree of accuracy. After some familiarity with this method is acquired, however, amplitude and angle plots for high-degree rational functions of $j\omega$ can be obtained more rapidly with this method than by analytical calculation or by factoring the polynomials and plotting with first- and second-degree templets. The inaccuracies usually occur in the upper frequency range. If such inaccuracies can be tolerated, the time required to obtain plots can be decreased considerably.

One possible method of decreasing the slopes of the individual terms of the real and imaginary parts of an nth degree polynomial might be to factor $(j\omega)^m$ from the polynomial where $m \approx \frac{n}{2}$. This process results in a modified polynomial whose real and imaginary parts contain terms of both positive and negative slopes when plotted logarithmically. After the amplitude and angle of this modified polynomial have been obtained, they must be corrected by the factor $(j\omega)^m$ to obtain the amplitude and angle of the original polynomial. Obtaining amplitude and phase of the modified polynomial by the method mentioned here would require term-pair templets representing terms having both positive and negative slopes. The possible advantages of this refinement have not been thoroughly investigated.

The use of logarithmic graph paper different from that used here may be desirable for many problems. For most problems, the use of logarithmic paper having more cycles in the vertical direction has been found advantageous. Templets are easily constructed for use with any logarithmic scale chosen (see appendix).

CONCLUDING REMARKS

A method has been presented for obtaining amplitude and phase plots for rational algebraic functions of an imaginary variable $F(j\omega)$. The method involves the use of templets but does not require the factoring of polynomials. After some familiarity with the method is acquired, amplitude and angle plots of high-degree rational functions of $j\omega$ can be obtained more rapidly with this method than by analytical calculation or by methods that involve the factoring of polynomials.

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National Advisory Committee for Aeronautics
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APPENDIX

FASHIONING THE TEMPLETS

The summing templet, the amplitude templet, and the angle scale each makes graphical use of two quantities q_1 and q_2 to obtain a desired result. It has been shown in the section entitled "Theoretical Analysis" that the use of either of these templets is independent of the magnitude of the individual quantities concerned but depends only on their ratio, that is, for example, the same points on the summing templet can be used to obtain $1 + 2 = 3$ or $3 + 6 = 9$ $\left(\text{since } \frac{6}{3} = \frac{2}{1}\right)$. Therefore, if one of the quantities q_1 is arbitrarily chosen to be unity, the values of the result $\left(q_1 + q_2\right.$ in the case of the summing templet$\left.\right)$ can be calculated and tabulated for discrete values of the second quantity q_2. The values so tabulated can then be used with the chosen logarithmic scale to make the templet. A tabulation of the values of $1 \pm q_2$ for use in making the summing templet is not necessary since these values are easily calculated mentally. A tabulation of the values of $\sqrt{1 + \left(q_2\right)^2}$ for use in making the amplitude templet is given in table II, and a tabulation of the values of q_2 against $\tan^{-1} \frac{q_2}{1}$ for use in making the angle scale is given in table III.

The exact shape of the contours of the summing templet or the amplitude templet is arbitrary, the only requirement being that the desired relationship between q_1, q_2, and $q_1 \pm q_2$ $\left(\text{or } \sqrt{\left(q_1\right)^2 + \left(q_2\right)^2}\right)$ be satisfied for all values of $\frac{q_2}{q_1}$ by the contours of the templets when applied to the chosen logarithmic scale. The general shape chosen here is convenient to use because the reference point can be anchored to a desired point on the graph with a sharp pointed instrument, and the templet can be rotated about this point to the desired position.

The term-pair templets all represent frequency functions of the form $\pm a\omega^n \pm b\omega^{n+2}$. As was pointed out in the section entitled "Theoretical Analysis," the shape of the logarithmic plot of such a function is independent of the magnitudes of a and b. Therefore, the shape of the contour representing such a function plotted to the chosen logarithmic scale can be determined by assuming $a = b = 1$. Values of $\pm a\omega^n \pm b\omega^{n+2}$ can then be tabulated for discrete values of ω for any desired value of n. Such a tabulation is given in table IV for $n = 0, 1, 4,$ and 5. The values tabulated are sufficient to make the four term-pair templets required for plotting polynomials of ninth degree. By plotting these values to the chosen logarithmic scale, the contours of the term-pair templets are obtained. These values are plotted in figure 2 to the logarithmic scale used in this report which is the scale of Keuffel & Esser Co. No. 359-111G logarithmic graph paper.

Satisfactory templets can be made from thin, transparent sheet plastic which can be cut with scissors and smoothed with a fine file.

REFERENCE

1. Brown, Gordon S., and Campbell, Donald P.: Principles of Servomechanisms. John Wiley & Sons, Inc., 1948.

TABLE I

PHASE ANGLE VALUES FOR ILLUSTRATIVE EXAMPLE

u (40 radians/sec)	$[\varphi(u)]_N$ (deg)	$[\varphi(u)]_D$ (deg)	$\varphi(u)$ (deg)	$\varphi(u)$ (deg) (a)
0.07	42	51	-9	-9.2
.08	46	55	-9	-9.7
.09	49	58.5	-9.5	-10.2
.1	52	62.5	-10.5	-10.8
.15	64	77	-13	-13.8
.2	71.5	87	-15.5	-15.5
.3	82	103	-21	-21.5
.4	-----	-----	------	-29.3
.5	93	132	-39	-39.2
.6	98	147.5	-49.5	-51.6
.7	102.5	166.5	-64	-66.1
.8	106.5	186	-80.5	-81.6
.9	111	206	-95	-96.3
1.0	114.5	225	-110.5	-108.8
1.1	119	239	-120	-119.0
1.2	123	252	-129	-127.3
1.3	128	265	-137	-134.3
1.4	132.5	277.5	-145	-140.5
1.5	138	289	-151	-146.2

[a]Calculated analytically.

TABLE II

VALUES FOR USE IN MAKING THE AMPLITUDE TEMPLET

q_2	$\sqrt{1 + (q_2)^2}$	q_2	$\sqrt{1 + (q_2)^2}$
1.0	1.414	5.6	5.689
1.2	1.562	5.8	5.886
1.4	1.720	6.0	6.083
1.6	1.887	6.2	6.280
1.8	2.059	6.4	6.478
2.0	2.236	6.6	6.675
2.2	2.417	6.8	6.873
2.4	2.600	7.0	7.071
2.6	2.786	7.2	7.269
2.8	2.973	7.4	7.467
3.0	3.162	7.6	7.655
3.2	3.353	7.8	7.864
3.4	3.544	8.0	8.062
3.6	3.736	8.2	8.261
3.8	3.829	8.4	8.459
4.0	4.123	8.6	8.658
4.2	4.317	8.8	8.857
4.4	4.512	9.0	9.055
4.6	4.707	9.2	9.254
4.8	4.903	9.4	9.453
5.0	5.099	9.6	9.652
5.2	5.295	9.8	9.851
5.4	5.492	10.0	10.05

TABLE III

VALUES FOR USE IN MAKING THE ANGLE SCALE

$\tan^{-1}q_2$ (deg)	q_2	$\tan^{-1}q_2$ (deg)	q_2	$\tan^{-1}q_2$ (deg)	q_2
1	0.0175	31	0.6009	61	1.804
2	.0350	32	.6249	62	1.881
3	.0524	33	.6494	63	1.963
4	.0699	34	.6745	64	2.050
5	.0875	35	.7002	65	2.144
6	.1051	36	.7265	66	2.246
7	.1228	37	.7536	67	2.356
8	.1405	38	.7813	68	2.475
9	.1584	39	.8098	69	2.605
10	.1763	40	.8391	70	2.748
11	.1944	41	.8693	71	2.904
12	.2126	42	.9004	72	3.078
13	.2309	43	.9325	73	3.271
14	.2493	44	.9657	74	3.487
15	.2680	45	1.000	75	3.732
16	.2868	46	1.036	76	4.011
17	.3057	47	1.072	77	4.332
18	.3249	48	1.111	78	4.705
19	.3443	49	1.150	79	5.145
20	.3640	50	1.192	80	5.671
21	.3839	51	1.235	81	6.314
22	.4040	52	1.280	82	7.115
23	.4245	53	1.327	83	8.144
24	.4452	54	1.376	84	9.514
25	.4663	55	1.428	85	11.43
26	.4877	56	1.483	86	14.30
27	.5095	57	1.540	87	19.08
28	.5317	58	1.600	88	28.64
29	.5543	59	1.664	89	57.29
30	.5774	60	1.732	90	0

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TABLE IV

VALUES FOR USE IN MAKING TERM-PAIR TEMPLETS

ω	$1 + \omega^2$	$1 - \omega^2$	$\omega + \omega^3$	$\omega - \omega^3$	$\omega^4 + \omega^6$	$\omega^4 - \omega^6$	$\omega^5 + \omega^7$	$\omega^5 - \omega^7$
0.1	1.010	0.990	0.101	0.099	0.0001	0.0001	0.00001	0.00001
.2	1.040	.960	.208	.192	.0017	.0015	.0003	.0003
.3	1.090	.910	.327	.273	.0088	.0074	.0026	.0022
.4	1.160	.840	.464	.336	.0297	.0215	.0118	.0086
.5	1.250	.750	.625	.375	.0781	.0469	.0391	.0235
.55	1.302	.698	.716	.384	.1192	.0638	.0656	.0351
.6	1.360	.640	.816	.384	.1763	.0829	.1058	.0498
.65	1.422	.578	.925	.375	.254	.1031	.1651	.0670
.7	1.490	.510	1.043	.357	.358	.1225	.250	.0857
.75	1.562	.438	1.172	.328	.494	.1384	.371	.1038
.80	1.640	.360	1.312	.288	.672	.1475	.537	.1180
.82	1.672	.328	1.371	.269	.756	.1481	.620	.1215
.84	1.706	.294	1.433	.247	.849	.1466	.713	.1231
.86	1.740	.260	1.496	.224	.952	.1424	.818	.1225
.88	1.774	.226	1.562	.1985	1.064	.1353	.936	.1191
.90	1.810	.1900	1.629	.1710	1.188	.1247	1.069	.1122
.92	1.846	.1536	1.699	.1413	1.323	.1100	1.217	.1012
.94	1.884	.1164	1.771	.1094	1.471	.0909	1.382	.0854
.96	1.922	.0784	1.845	.0753	1.632	.0666	1.567	.0639
.98	1.960	.0396	1.921	.0388	1.808	.0365	1.772	.0358
1.00	2.00	0	2.00	0	2.00	0	2.00	0
1.05	2.10	-.1025	2.21	-.1076	2.56	-.125	2.68	-.131
1.1	2.21	-.210	2.43	-.231	3.24	-.307	3.56	-.338
1.2	2.44	-.440	2.93	-.528	5.06	-.912	6.07	-1.095
1.3	2.69	-.690	3.50	-.897	7.68	-1.971	9.99	-2.56
1.4	2.96	-.960	4.14	-1.344	11.37	-3.69	15.92	-5.16
1.5	3.25	-1.250	4.88	-1.875	16.45	-6.33	24.7	-9.49
1.6	3.56	-1.560	5.70	-2.50	23.3	-10.22	37.3	-16.36
1.7	3.89	-1.890	6.61	-3.21	32.5	-15.79	55.2	-26.8
1.8	4.24	-2.24	7.63	-4.03	44.5	-23.5	80.1	-42.3
1.9	4.61	-2.61	8.76	-4.96	60.1	-34.0	114.1	-64.6
2.0	5.00	-3.00	10.00	-6.00	80.0	-48.0	160.0	-96.0
2.5	7.25	-5.25	18.12	-13.12	283	-205	708	-513
3.0	10.00	-8.00	30.0	-24.0	810	-648	2430	-1944
3.5	13.25	-11.25	46.4	-39.4	1988	-1688	6959	-5909
4.0	17.00	-15.00	68.0	-60.0	4352	-3840	17408	-15360

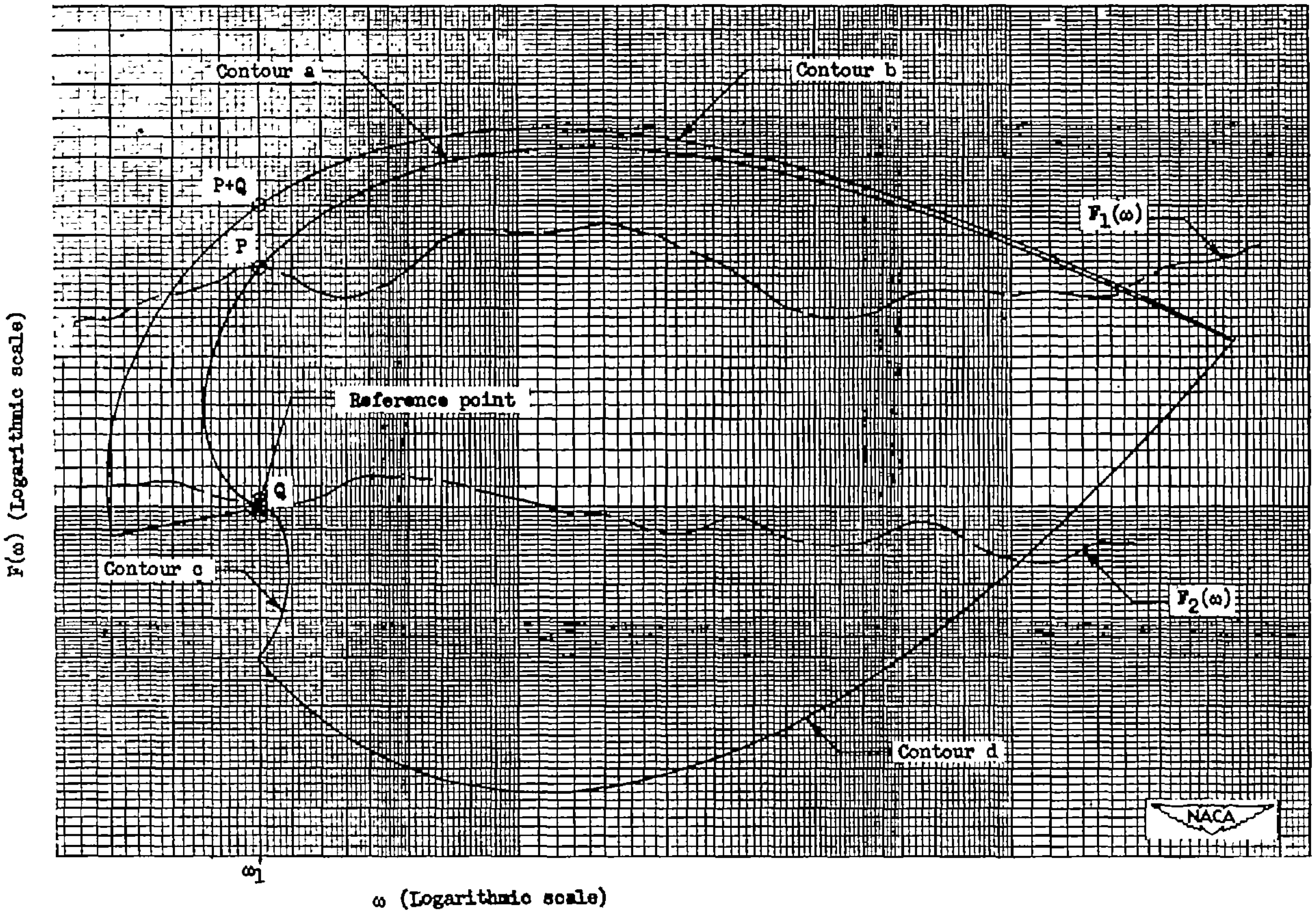
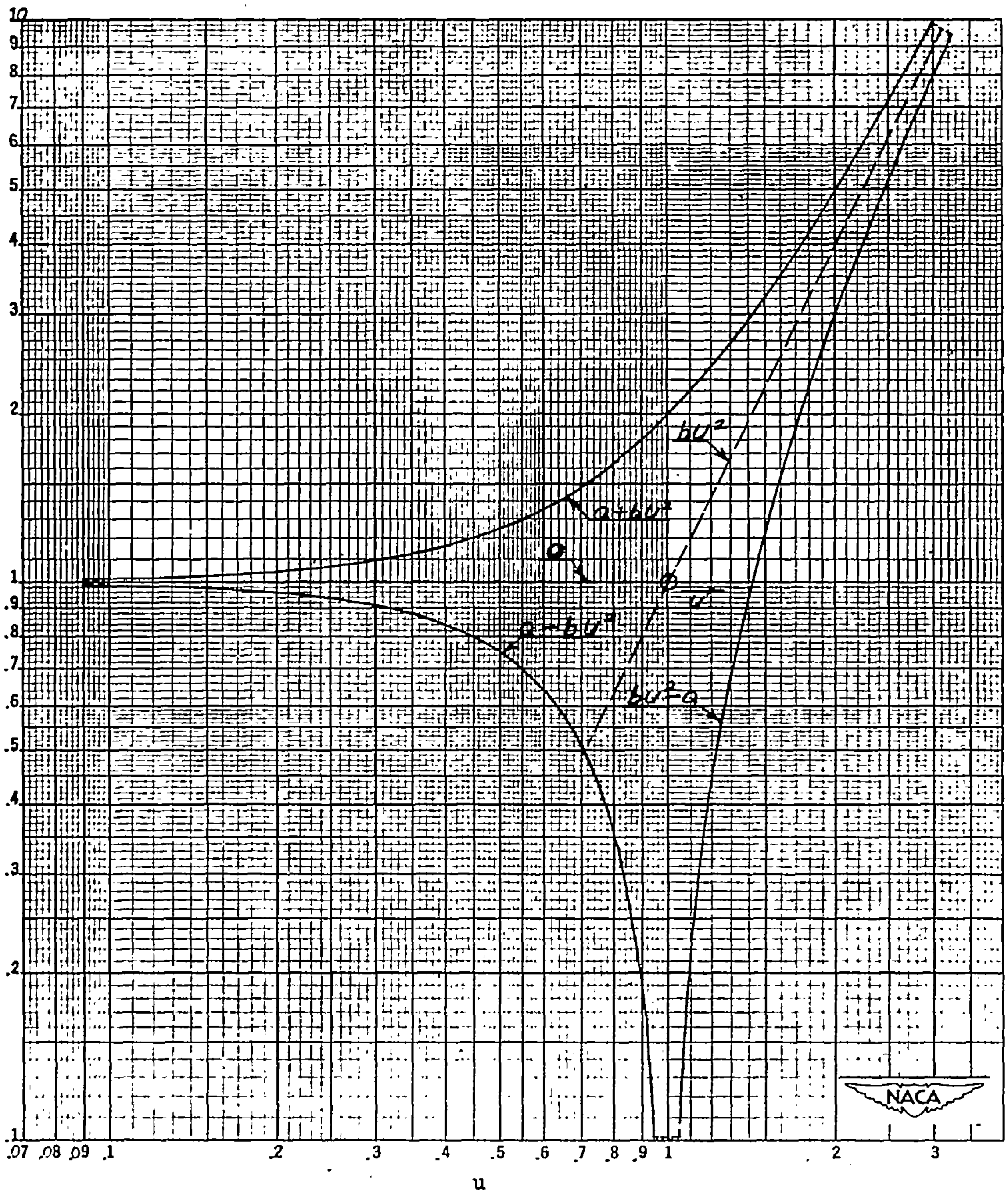
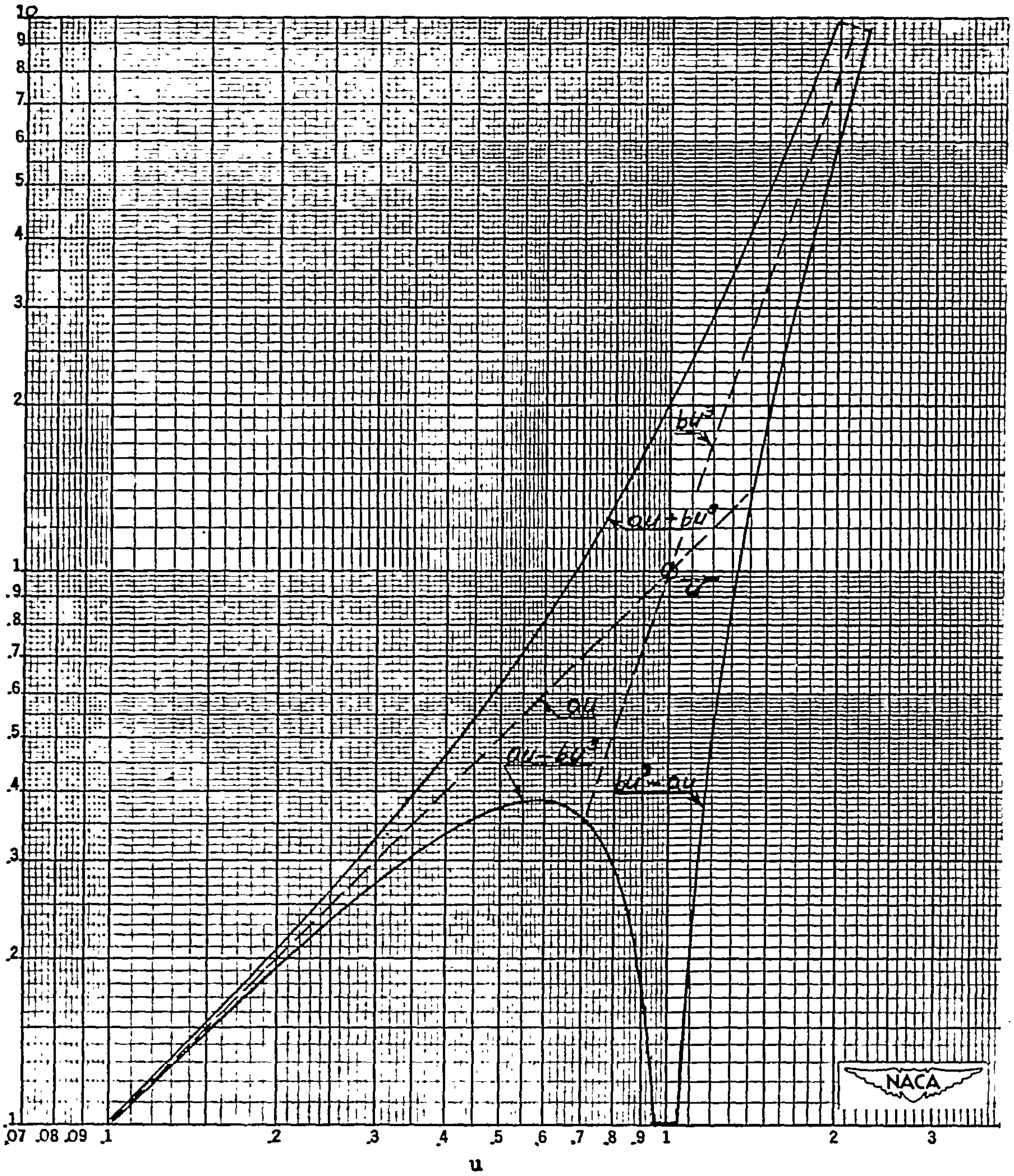


Figure 1.- Summing templet with illustration of its use.



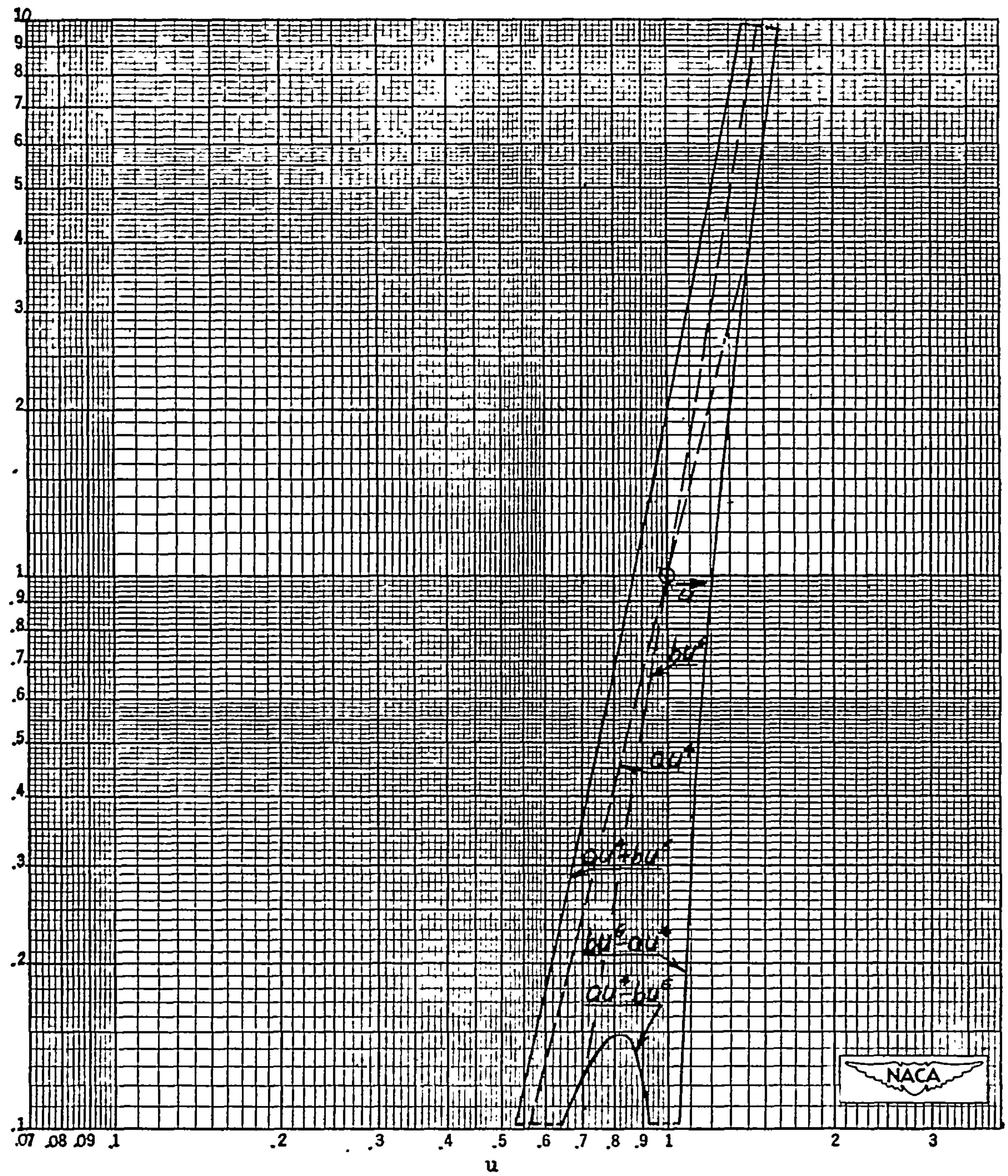
(a) Contours of term-pair templets representing logarithmic plots of $\pm a \pm bu^2$, shown oriented for $a = b = 1$.

Figure 2.- Term-pair contours.



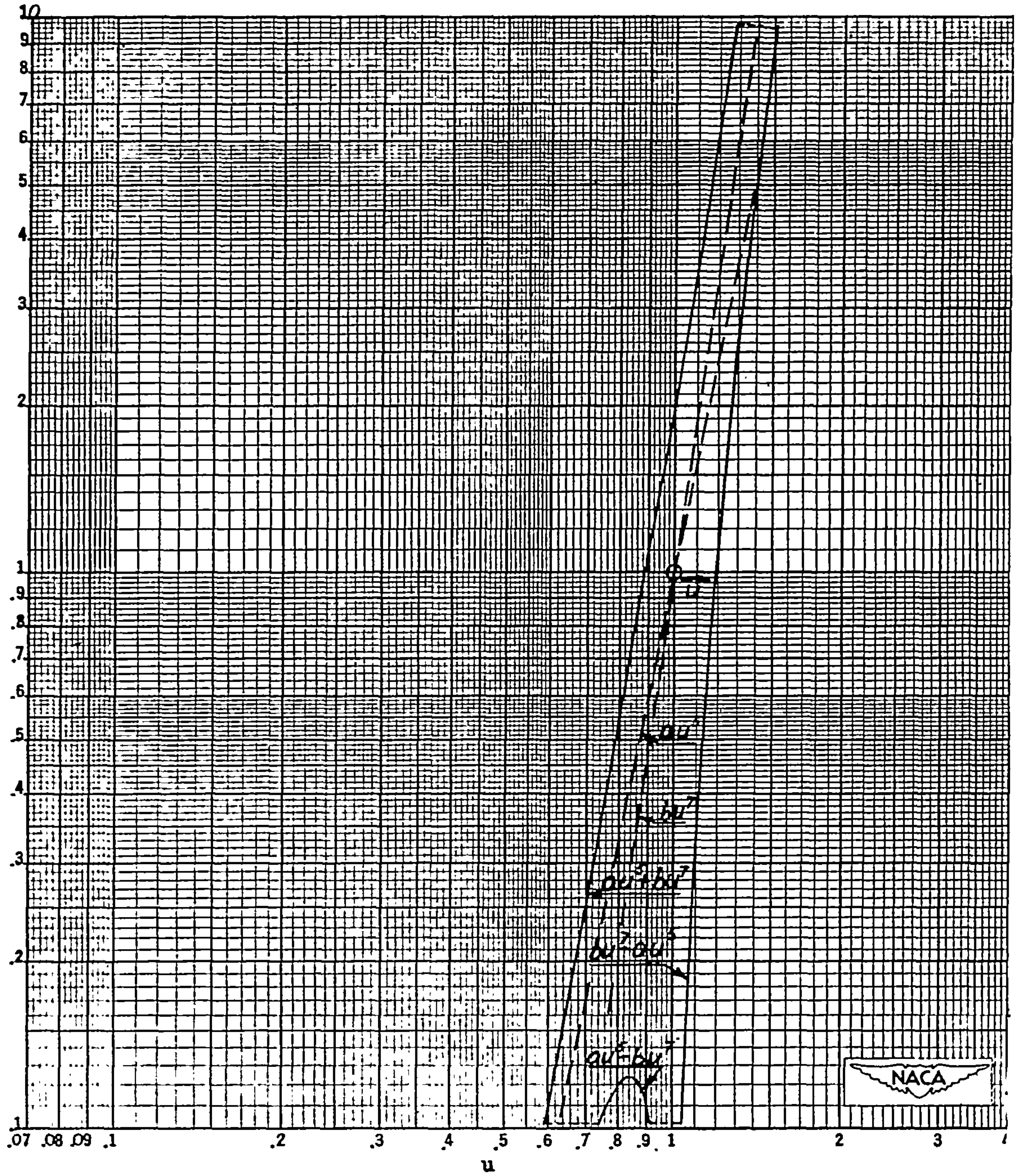
(b) Contours of term-pair templets representing logarithmic plots of $\pm au \pm bu^3$, shown oriented for $a = b = 1$.

Figure 2.- Continued.



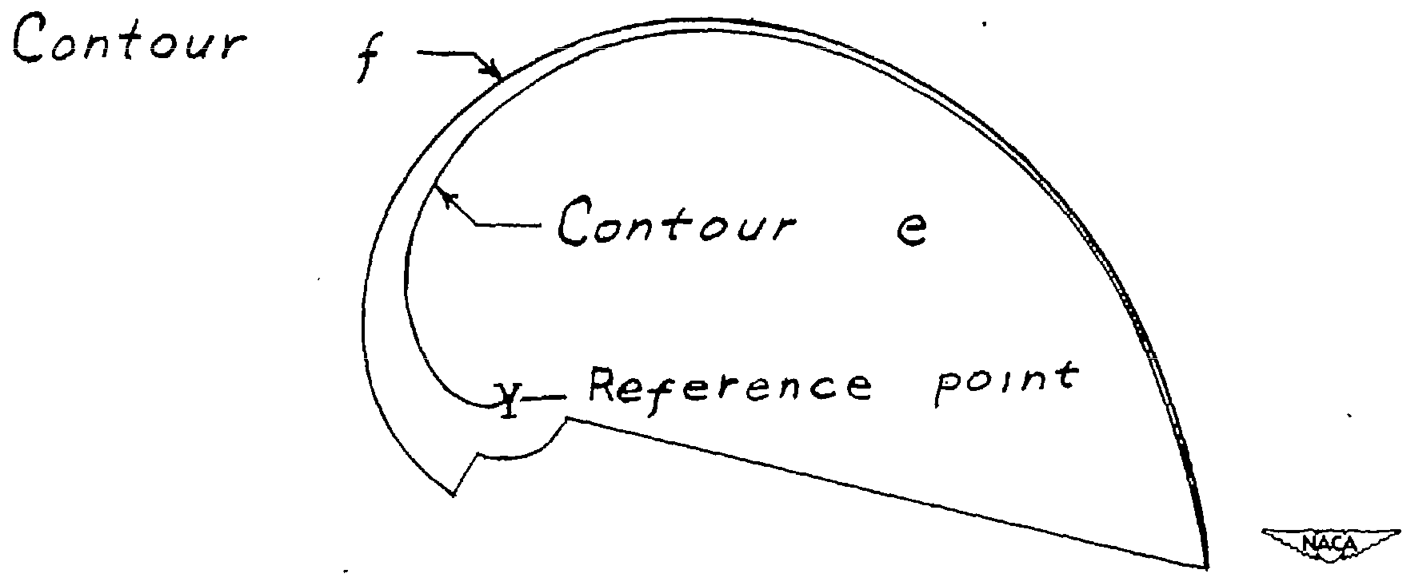
(c) Contours of term-pair templets representing logarithmic plots of $\pm au^4 \pm bu^6$, shown oriented for $a = b = 1$.

Figure 2.- Continued.

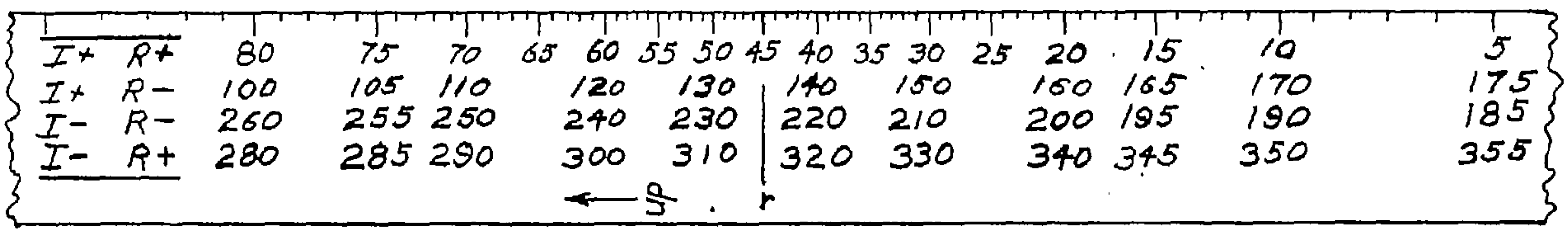


(d) Contours of term-pair templets representing logarithmic plots of $\pm au^5 \pm bu^7$, shown oriented for $a = b = 1$.

Figure 2.- Concluded.



(a) Amplitude templet.



(b) Angle scale.

Figure 3.- Amplitude templet and angle scale.

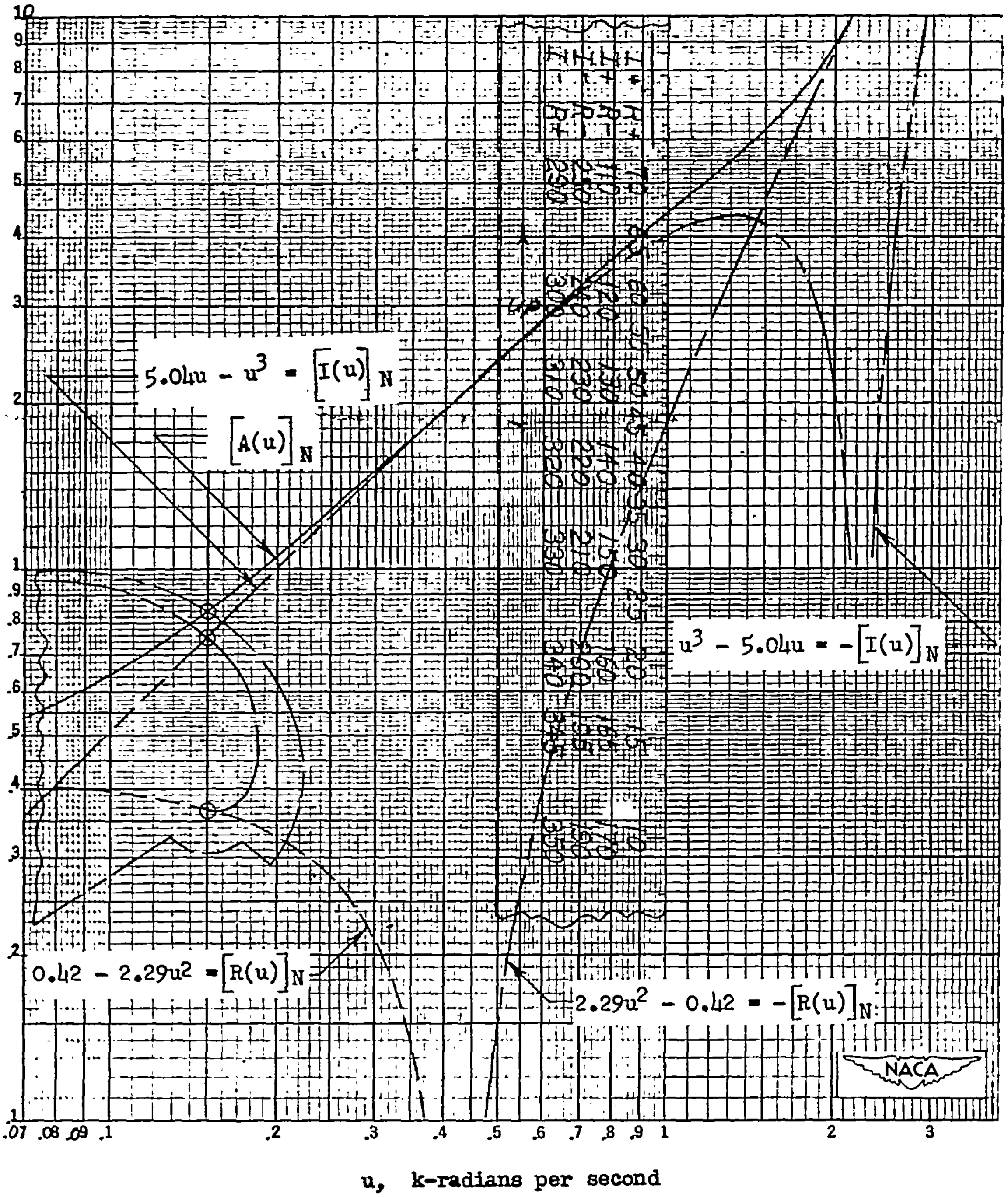


Figure 4.- Logarithmic plots of $\pm[R(u)]_N$, $\pm[I(u)]_N$, and $[A(u)]_N$ for illustrative example showing the use of the amplitude templet and the angle scale.

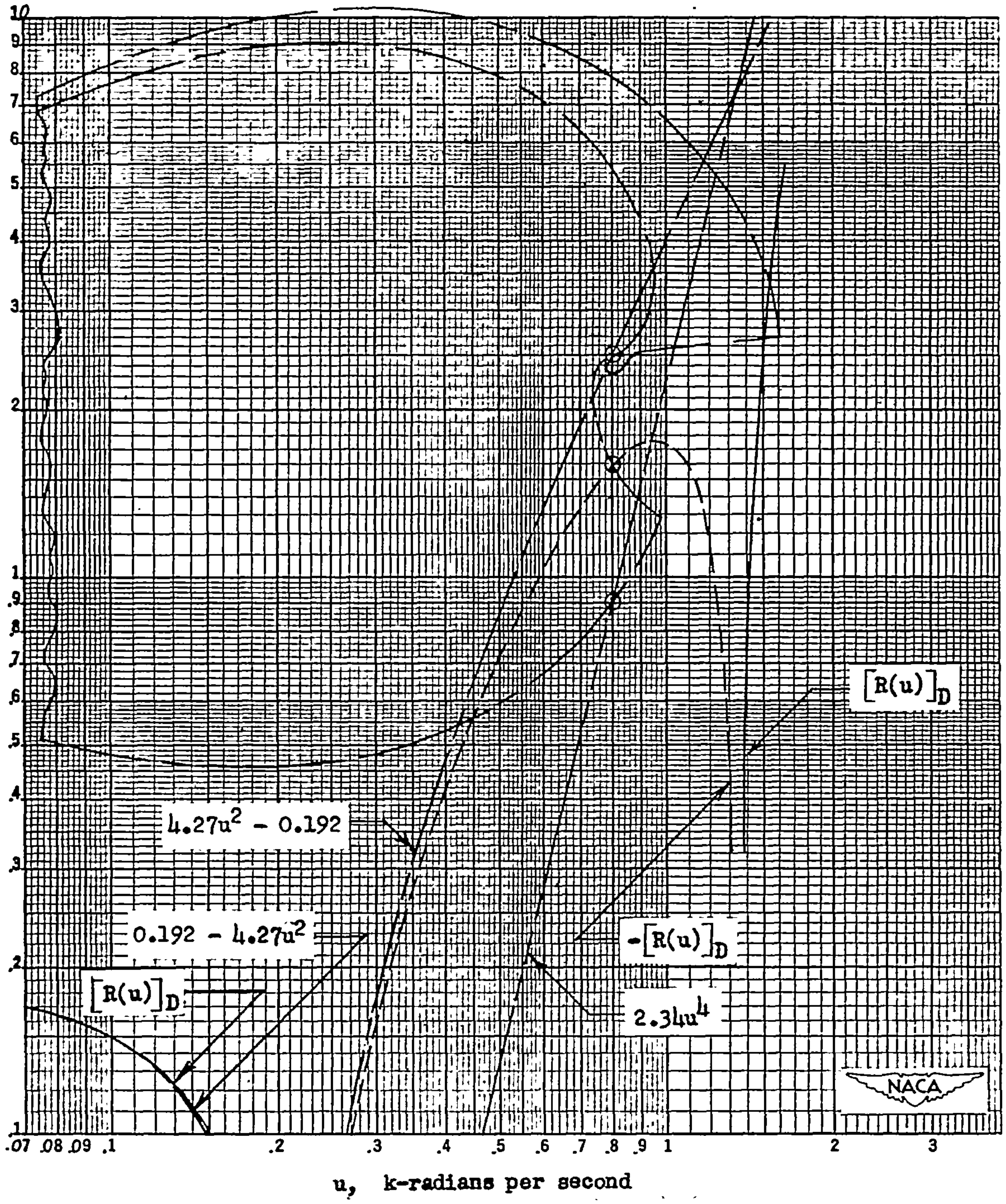


Figure 5.- Logarithmic plots of $\pm(0.192 - 4.27u^2)$, $2.34u^4$, and $\pm[R(u)]_D$ for illustrative example showing the use of the summing templet for subtraction.

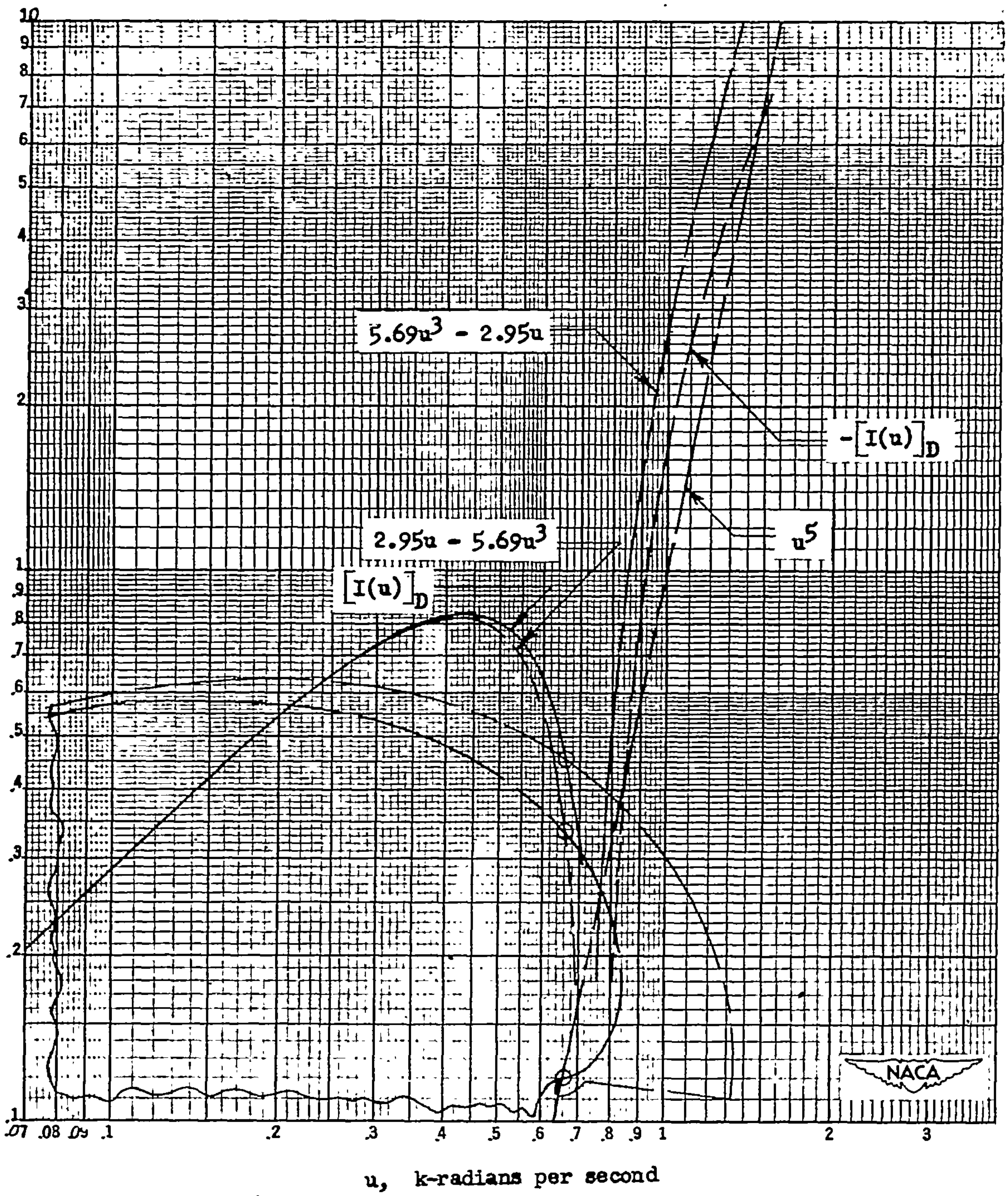


Figure 6.- Logarithmic plots of $\pm(2.95u - 5.69u^3)$, u^5, and $\pm[I(u)]_D$ for illustrative example showing the use of the summing templet for addition.

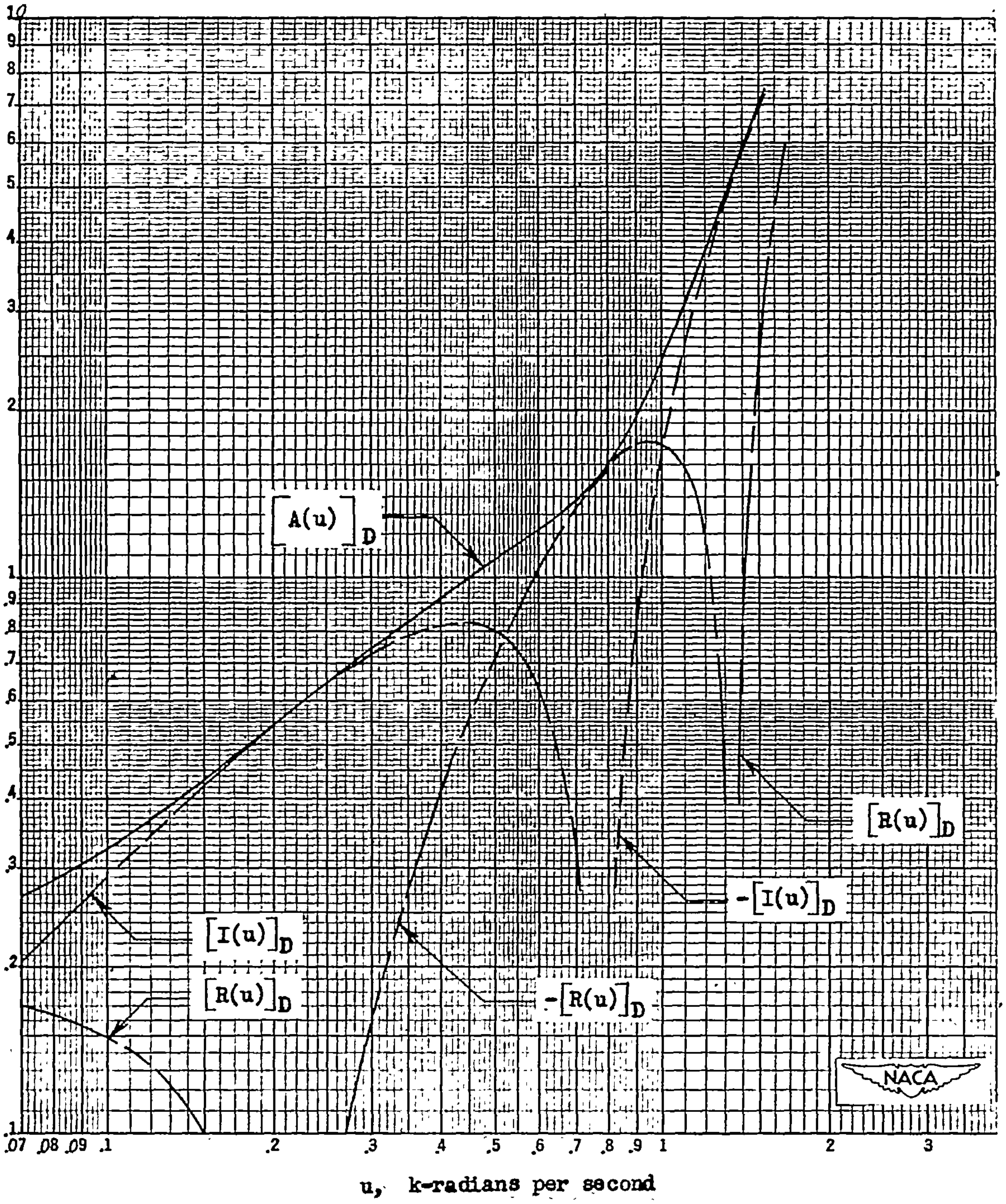


Figure 7.- Logarithmic plots of $\pm\left[R(u)\right]_D$, $\pm\left[I(u)\right]_D$, and $\left[A(u)\right]_D$ for illustrative example.

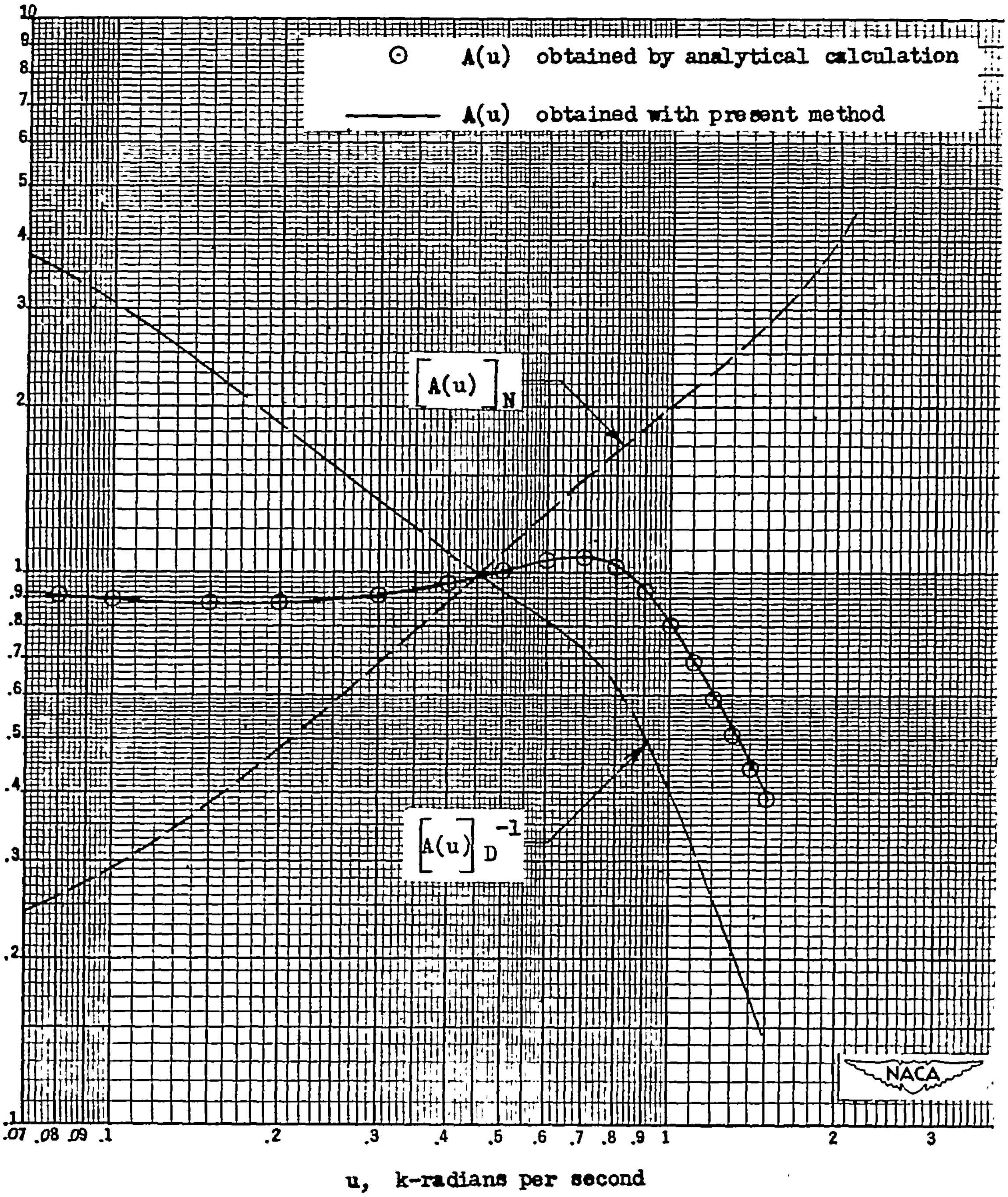


Figure 8.- Logarithmic plots of $\left[A(u)\right]_N$, $\left[A(u)\right]_D^{-1}$, and $A(u)$ for illustrative example.

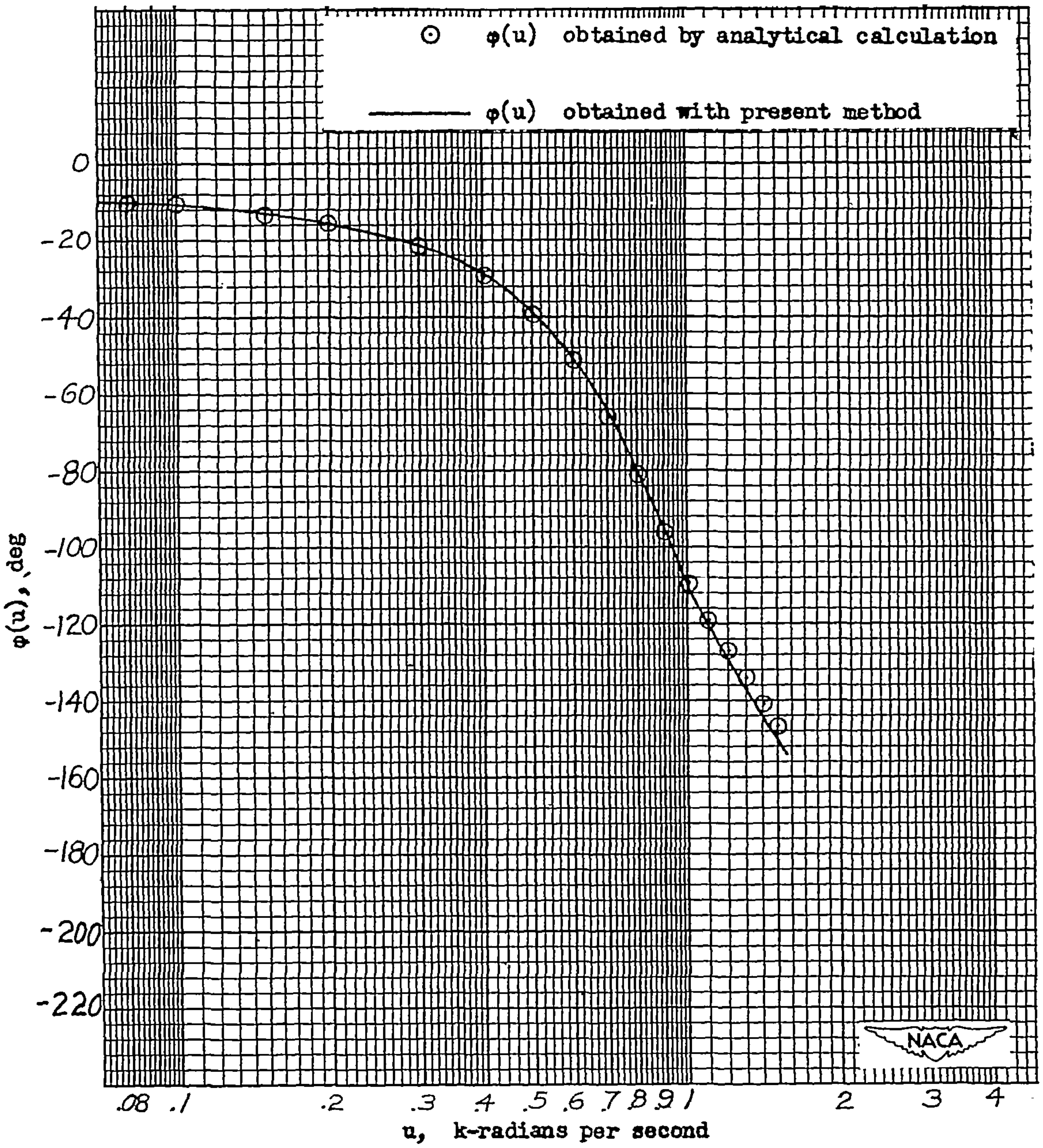


Figure 9.- Plot of $\varphi(u)$ for illustrative example.

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